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The Small Scale Structure of Space-Time: A Bibliographical Review

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Abstract

This essay is a tour around many of the lesser known pregeometric models of physics, as well as the mainstream approaches to quantum gravity, in search of common themes which may provide a glimpse of the final theory which must lie behind them.

Introduction

In this review I will reflect on some of the developments in quantum gravity which have emerged over the last 15 years. It is remarkable that these have advanced despite the lack of any experimental input at anywhere near the relevant energies. The theories are built on purely mathematical foundations, turning the clock back past 2,500 years of scientific method to emulate the ancient methodology of Plato. It might well be asked in view of this if physicists are accomplishing anything more than constructing better Platonic solids.

But if they were working on such shaky ground we might expect to see a collection of mutually incompatible ideas emerging. What encourages us to believe that something related to real physics is being studied is that the various threads of development show signs of deep connections suggesting that they could all turn out to be aspects of some underlying unified theory.

Many theoretical physicists believe that to progress much further it will be necessary to rethink our understanding of space-time. The 4D manifold structure of general relativity does not seem adequate to describe the kind of processes which are implicated in quantum gravity. This poses a difficult puzzle. Can we be hopeful that the necessary mathematics to describe physics beyond the Planck scale is within our understanding?

We can at least be optimistic that experimental physics has more to tell us. Future accelerators may find supersymmetry or something else unexpected and lead us to the correct unified theory at the GUT scale. It may then be possible to demonstrate that this model is an unfalsifiable low energy limit of some more fully unified model such as string theory and in the absence of other plausible models most people will accept its validity. But that is not the end of the story. String theory lacks predictive power at high energies where it becomes interesting. To go beyond this point it will be necessary to understand the nature of space-time itself. Experiment is unlikely to provide any direct help since the appropriate energies are way beyond reach.

The outstanding problem in theoretical physics today is to uncover the mathematical origins of string theory which would explain its enigmatic properties and allow us to solve it. I contend that the key discovery is that the symmetry of space-time must be extended to one which is event-symmetric.

As well as giving potted reviews of the mainstream theories of Quantum Gravity, I have collected together here a number of the better theoretical

ideas which have been proposed as ways to go beyond our current understanding of the structure of space-time and replace it with some form of pregeometry. These ideas have been developed largely in isolation and from a variety of motivations. At present they are at the fringe of mainstream physics. I find however that many common ideas have emerged and that considerable mathematical maturity has been introduced into the field in recent years.

20 years ago there were very few physicists who studied the small scale structure of space-time. Among those who did the names of Wheeler [1] and Finkelstein [2] stand out as two who independently conceived many of the foundation principles. They were then ahead of their time but now things have progressed. We are beginning to understand enough about quantum gravity to gain the necessary physical insight and with new mathematical tools such as quantum groups we may have what we need to tackle the problem. Against the odds it now starts to look as if we have a chance to reach some understanding of space-time beyond the traditional continuum model.

An important part of this paper is the list of references which includes many articles on discrete models of space-time as well as many others which provide clues and motivation. With the aid of the QSPIRES database in particular I have constructed a diverse list although it is by no means complete. Sadly many of the more speculative ideas seem only to have appeared as preprints which are never published or have been presented at conferences and it is difficult to trace the proceedings some years later on. Hopefully the introduction of electronic pre-print archives will enable such documents to remain accessible in future. My treatment of the work in the references is necessarily shallow since there is a larger quantity than might have been imagined. The reader will have to consult the original articles in order to properly appreciate their significance.

A part of this essay is devoted to my own work on models of event-symmetric space-time. These models demonstrate a way in which string field theories might be formulated in a geometric non-perturbative setting with all its symmetry manifest. Topology change, mirror manifolds and duality in string theory all suggest that diffeomorphism invariance is too limited a symmetry for string theory. The event-symmetric theory may be part of the solution to this problem.

This paper grew out of my own notes in which I have tried to collect the pieces of a global picture of the theory to which quantum gravity research is leading. I am probably not the right person to provide such a general

review since my general understanding of these topics remains fairly shallow. However, I have found a great deal of enjoyment in comparing my own ideas and motivations with those of others and hope that this document will help introduce some others to research on the small-scale structure of space-time.

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Is Space-Time Discrete?

At a seaport in the Aegean around the year 500BC two philosophers, Leucippus and his student Democritus, pondered the idea that matter was made of indivisible units separated by void. Was it a remarkable piece of insight or just a lucky guess? At the time there was certainly no compelling evidence for such a hypothesis. Their belief in the atom was a response to questions posed earlier by Parmenides and Zeno. Perhaps they were also inspired by the coarseness of natural materials like sand and stone. Democritus extended the concept as far as it could go claiming that not just matter, but everything else from colour to the human soul must also consist of atoms [3].

The idea was subsequently supplanted by the very different philosophies of Plato and Aristotle who believed that matter was infinitely divisible and that nature was constructed from perfect symmetry and geometry. It was not until the eighteenth and nineteenth centuries that the atomic theory was resurrected by Dalton, Maxwell and Boltzman. This time they had better scientific grounds to support their belief. They were able to explain pressure, heat and chemical reactions in terms of interactions between atoms and molecules. Despite this indirect evidence the majority of scientists disfavoured the theory until Einstein explained that Brownian motion could be seen as direct experimental evidence of molecular motions. But how far has modern physics gone towards the ideal of Democritus that everything should be discrete?

The story of light parallels that of matter. It was Newton who first championed the corpuscular theory of light but without good foundation. Everything he had observed and much more was explained by Maxwell's theory of Electromagnetism in terms of waves in continuous fields. It was Planck's Law and the photoelectric effect which later upset the continuous theory. These phenomena could best be explained in terms of light quanta. Today we can detect the impact of individual photons on CCD cameras even after they have travelled across most of the observable universe from the earliest moments of galaxy formation.

Those who resisted the particle concepts had, nevertheless, some good sense. Light and matter, it turns out, are both particle and wave at the same time. The paradox is resolved within the framework of Quantum Field Theories where the duality arises from different choices of basis in the Hilbert space of the wave function.

After matter and light, history is repeating itself for a third time and

now it is space-time which is threatened to be reduced to discrete events. The idea that space-time could be discrete has been a recurring one in the scientific literature. A survey of just a few examples reveals that discrete space-time can actually mean many things and is motivated by a variety of philosophical or theoretical influences.

It has been apparent since early times that there is something different between the mathematical properties of the real numbers and the quantities of measurement in physics at small scales. Riemann himself remarked on this disparity even as he constructed the formalism which would be used to describe the space-time continuum for the next century of physics [4]. When you measure a distance or time interval you can not declare the result to be rational or irrational no matter how accurate you manage to be. Furthermore it appears that there is a limit to the amount of detail contained in a volume of space. If we look under a powerful microscope at a grain of dust we do not expect to see minuscule universes supporting the complexity of life seen at larger scales. Structure becomes simpler at smaller distances. Surely there must be some minimum length at which the simplest elements of natural structure are found and surely this must mean space-time is discrete.

This style of argument tends to be convincing only to those who already believe the hypothesis. It will not make many conversions. After all, the modern formalism of axiomatic mathematics leaves no room for Zeno's paradox of Achilles and the tortoise. However, such observations and the discovery of quantum theory with its discrete energy levels [5] and the Heisenberg uncertainty principle [6] led physicists to speculate that space-time itself may be discrete as early as the 1930's [7, 8]. In 1936 Einstein expressed the general feeling that "*... perhaps the success of the Heisenberg method points to a purely algebraic method of description of nature, that is, to the elimination of continuous functions from physics. Then, however, we must give up, by principle, the space-time continuum ...*" [9]. Heisenberg himself noted that physics must have a fundamental length scale which together with Planck's constant and the speed of light permit the derivation of particle masses [10, 11]. Others also argued that it would represent a limit on the measurement of space-time distance [12]. At the time it was thought that this length scale would be around $10^{-15}m$ corresponding to the masses of the heaviest particles known at the time but searches for non-local effects in high energy particle collisions have given negative results for scales down to about $10^{-19}m$ [13] and today the consensus is that it must correspond to the much smaller Planck length at $10^{-35}m$ [14].

The belief in some new space-time structure at small length scales was reinforced after the discovery of ultraviolet divergences in Quantum Field Theory. Even though it was possible to perform accurate calculations by a process of renormalisation [15, 16, 17, 18] many physicists felt that the method was incomplete and would break down at smaller length scales unless a natural cutoff was introduced.

A technique which introduces such a minimum length into physics by quantising space-time was attempted by Snyder in 1947 [19, 20]. Snyder introduced non-commutative operators for space-time coordinates. These operators have a discrete spectrum and so lead to a discrete interpretation of space-time. The model was Lorentz invariant but failed to preserve translation invariance. Similar methods have been tried by others since [21, 22, 23, 24, 25, 26, 27, 28]. The quantisation results in differential operators being replaced by finite difference operators as if they were acting on a discrete space-time. The momentum space is therefore compact. An alternative way to get a similar effect is to start from a momentum space which has a constant curvature [29, 30]. Although no complete theory has come of these ideas there has been a recent upsurge of renewed interest in quantised space-time, now reexamined in the light of quantum groups. We will return to this later. Another modern approach to quantised space-time is provided by Prugovecki [31, 32].

Quite a number of alternatives and variations on quantised space-time have been tried over the years. Yukawa and Heisenberg and others considered non-local models or field theory of particles which were not pointlike [10, 33, 34, 35, 11, 36, 37]. Similarly again, the minimum length can be introduced by stochastically averaging over a small volume of space-time [38, 39, 40, 41]. The bane of all these models is loss of causality. We might regard superstring theory as the eventual successful culmination of this program [25] since it describes a field theory of non-point like objects which respect causality.

Another way to provide a small distance cut-off in field theory is to formulate it on a discrete lattice. This approach was also introduced early by Wentzel [42] but only later studied in any depth [43, 44, 45, 46, 47, 48]. Most recently lattice models of space-time have been studied by Yamamoto et al [49, 50, 51] and Preparata et al [52, 53, 54]. If the continuum limit is not to be restored by taking the limit where the lattice spacing goes to zero then the issue of the loss of Lorentz invariance must be addressed [55, 56]. For some authors a space-time in which the coordinates take rational values can be called discrete [57, 58, 59]. Lorentz invariance is then possible but

there is no minimum length scale.

None of these ideas were really very inventive in the way they saw space-time. Only a rare few such as Finkelstein with his space-time code [60, 61, 62, 63, 64] or Penrose with twistor theory [65] and spin networks [66] could come up with any concrete suggestions for a more radical pregeometry before the 1980's.

Another aspect of the quantum theory which caused disquiet was its inherent indeterminacy and the essential role of the observer in measurements. The Copenhagen interpretation seemed inadequate and alternative hidden variable theories were sought. It was felt that quantum mechanics would be a statistical consequence of a more profound discrete deterministic theory in the same sense that thermodynamics is a consequence of the kinetic gas theory.

Over the years many of the problems which surrounded the development of the quantum theory have diminished. Renormalisation itself has become acceptable and is proven to be a consistent procedure in perturbation theory of gauge field physics [67]. The perturbation series itself may not be convergent but gauge theories can be regularised non-perturbatively on a discrete lattice [47] and there is good reason to believe that consistent Quantum Field Theory can be defined on continuous space-time at least for non-abelian gauge theories which are asymptotically free [68]. In Lattice QCD the lattice spacing can be taken to zero while the coupling constant is rescaled according to the renormalisation group [69]. In the continuum limit there are an infinite number of degrees of freedom in any volume no matter how small. The Nielsen-Ninomiya no-go theorem [70, 71] spells a problem for the inclusion of fermions but this too may be possible to resolve [72, 73].

Quantum indeterminacy has also become an acceptable aspect of physics. Everett's thesis which leads us to interpret quantum mechanics as a realisation of many worlds [74] has been seen as a resolution of the measurement problem for much of the physics community.

Without the physical motivation discrete space-time is disfavoured by many. Hawking says *"Although there have been suggestions that space-time may have a discrete structure I see no reason to abandon the continuum theories that have been so successful"* [75]. Hawking makes a valid point but it may be possible to satisfy everyone by invoking a discrete structure of space-time without abandoning the continuum theories if the discrete-continuum duality can be resolved as it was for light and matter.

Discreteness in Quantum Gravity

It is only when we try to include gravity in Quantum Field Theory that we find solid reason to believe in discrete space-time. With quantisation of gravity all the old renormalisation issues return and many new problems arise [76]. Whichever approach to quantum gravity is taken the conclusion seems to be that the Planck length is a minimum size beyond which the Heisenberg Uncertainty Principle prevents measurement if applied to the metric field of Einstein Gravity [77]. This can be expressed in a generalised uncertainty principle [78, 79].

Does this mean that space-time is discrete at such scales with only a finite number of degrees of freedom per unit volume? Recent theoretical results from String Theories and the Loop-representation of Gravity do suggest that space-time has some discrete aspects at the Planck scale [80, 81, 82, 83].

The far reaching work of Bekenstein and Hawking on black hole thermodynamics [84, 85, 86, 87] has led to some of the most compelling evidence for discreteness at the Planck scale. The *black hole information loss paradox* [88] which arises from semi-classical treatments of quantum gravity is the nearest thing physicists have to an experimental result in quantum gravity. Its resolution is likely to say something useful about a more complete quantum gravity theory. There are several proposed ways in which the paradox may be resolved most of which imply some problematical breakdown of quantum mechanics [89] while others lead to seemingly bizarre conclusions.

One approach is to suppose that no more information goes in than can be displayed on the event horizon and that it comes back out as the black hole evaporates by Hawking radiation. Bekenstein has shown that if this is the case then very strict and counter-intuitive limits must be placed on the maximum amount of information held in a region of space [90, 91]. It has been argued by 't Hooft that this finiteness of entropy and information in a black-hole is also evidence for the discreteness of space-time [92]. In fact the number of degrees of freedom must be given by the area in Planck units of a surface surrounding the region of space. This has led to some speculative ideas about how quantum gravity theories might work through a holographic mechanism [93, 94], i.e. it is suggested that physics must be formulated with degrees of freedom distributed on a two dimensional surface with the third spatial dimension being dynamically generated.

At this point it may be appropriate to discuss the prospects for experimental results in quantum gravity and small scale space-time structure. Over the past twenty years or more, experimental high energy physics has

mostly served to verify the correctness of the Standard Model as proposed theoretically in 1967 [95]. We now have theories extending to energies way beyond current accelerator technology but it should not be forgotten that limits set by experiment have helped to narrow down the possibilities and will presumably continue to do so.

It may seem that there is very little hope of any experimental input into quantum gravity research because the Planck energy is so far beyond reach. However, a theory of quantum gravity would almost certainly have low energy consequences which may be in reach of future experiments. The discovery of supersymmetry, for example, would have significant consequences for theoretical research on space-time structure.

It seems unlikely that any experiment short of studying the death throes of a small black hole in the laboratory can give direct support for quantum gravity research or fine structure of space-time. A number of possible signatures of quantum gravity have been identified [96] and there is, controversially, some hope that effects may be observable in realistic experiments see e.g. [97, 98, 99, 100, 101, 102].

It from Bit and the Theory of Theories

In the late 1970's the increasing power of computers seems to have been the inspiration behind some new discrete thinking in physics. Monte Carlo simulations of lattice field theories were found to give useful numerical results with surprisingly few degrees of freedom where analytic methods have made only limited progress.

Cellular automata (see [103]) became popular at the same time with Conway's invention of the Game of Life. Despite its simple rules defined on a discrete lattice of cells the game has some features in common with the laws of physics. There is a maximum speed for causal propagation which plays a role similar to the speed of light in special relativity. Even more intriguing is the accidental appearance of various species of "glider" which move through the lattice at fixed speeds. These could be compared with elementary particles.

For those seeking to reduce physics to simple deterministic laws this was an inspiration to look for cellular automata as toy models of particle physics despite the obvious flaw that they broke space-time symmetries [104, 105, 106, 107, 108]. Nevertheless the quest is not completely hopeless. With some reflection it is realised that a simulation of an Ising model [109, 110] with

a metropolis algorithm is a cellular automaton if the definition is relaxed to allow probabilistic transitions. The Ising model has a continuum limit in which rotational symmetry is restored. It is important to our understanding of integrable quantum field theories in two dimensions. Other extended cellular automata can be used to model fermions [111]. t'Hooft has also looked to cellular automata as a model of discrete space-time physics [112, 113, 114, 93]. His motivation is somewhat different since indeterminacy in quantum mechanics is, for him, quite acceptable. He suggests that the states of a cellular automata could be seen as the basis of a Hilbert space on which quantum mechanics is formed.

The influence of computers in physics runs to deeper theories than cellular automata. There is a school of thought which believes that the laws of physics will ultimately be understood as being a result from information theory. The basic unit of information is the binary digit or bit and the number of bits of information in a physical system is related to its entropy.

J.A Wheeler [115] has sought to extend this idea, “.. *every physical quantity, every it, derives its ultimate significance from bits, a conclusion which we epitomise in the phrase, It from Bit*”. For Wheeler and his followers the continuum is a myth, but he goes further than just making space-time discrete. Space-time itself, he argues, must be understood in terms of a more fundamental *pregeometry* [116, 117, 118]. In the pregeometry there would be no direct concepts of dimension or causality. Such things would only appear as emergent properties in the space-time idealisation.

The pregeometry precept rings true to many physicists and even underlies many attempts to understand the deeper origins of string theory. As Green puts it “*In string theory there aren't four or ten dimensions. That's only an approximation. In the deeper formulation of the theory the whole notion of what we mean by a dimension of space-time will have to be altered.*” [119].

Wheeler gives just a few clues as to how a pregeometry might be formulated of which the most concrete is the principle that the boundary of a boundary is zero [120]. This is a central result from algebraic topology which has become significant in non-commutative geometry.

The history of theoretical Physics has been a succession of reductions to lower levels, more fundamental, more unified, more symmetrical and ideally simpler. There is a strong belief that this process must eventually finish [121] but with what? According to *It from Bit* the process will bottom out in some principle of information theory. A skeptic would demand how a single mathematical principle could be so important as to spontaneously

bring about the existence of the universe while others fail to do so. And how are we to explain the importance of symmetry and the unreasonable effectiveness of mathematics in physics as demanded by Wigner [122]?

The answers may lie in an understanding of algorithmic complexity. For centuries mathematicians have looked at specific structures with simple definitions and interesting behaviour. With the advent of powerful computers they are beginning to look at general behaviour of complex systems. It was Feigenbaum who made the discovery that complex systems of chaotic non-linear equations exhibit a universal renormalisation behaviour characterised by the Feigenbaum constants [123, 124]. This type of universality has an independent existence which transcends details of the specific equations which generate it.

If we wish to understand the origins and meaning of physical law we may need to recognise them as the universal behaviour of a very general class of complex systems. Algorithmic information theory is perhaps a good place to look because computability is a universal property independent of the programming language syntax used to define it. In the statistical physics of systems with a large number of degrees of freedom we also find that the laws of thermodynamics emerge as a universal behaviour independent of microscopic details. Could we apply statistical methods to the general behaviour of algorithms?

We know from Feynman's Path Integral formulation of quantum mechanics that the evolution of the universe can be understood as a supposition of all possible histories that it can follow classically. The expectation values of observables are dominated by a small subset of possibilities whose contributions are reinforced by constructive interference. The same principle is at work in statistical physics where a vast state space is dominated by contributions at maximum entropy leading to thermodynamic behaviour. We might well ask if the same can be applied to mathematical systems in general to reveal the laws of physics as a universal behaviour which dominates the space of all possible theories and which transcends details of the construction of individual theories. If this was the case then we would expect the most fundamental laws of physics to have many independent formulations with no one of them standing out as the simplest. This might be able to explain why such a large subset of mathematics is so important in physics.

This philosophy is known as the Theory of all Theories. In general it is rather hard to make progress since it would be necessary to define an appropriate topology and measure in the space of all theories. It should be possible to get away with some reasonable subset of theories which forms a

dense covering of the topology so that it has, in some sense, arbitrarily good approximations to any significant point. In a restricted form where the subset comprising all 2 dimensional conformal field theories is taken, there has been some qualified success in understanding the non-perturbative origins of string theories [125, 126, 127, 128]. It is found that the renormalisation group flow in the space of theories converges at fixed points which may indicate the true vacuum of string theory.

Can we use the Theory of all Theories to explain why symmetry is so important in physics? There is a partial answer to this question which derives from an understanding of critical behaviour in statistical physics. Consider a lattice approximation to a Yang-Mills quantum field theory in the Euclidean sector. The Wilson discretisation preserves a discrete form of the gauge symmetry but destroys the space-time rotational symmetry. If we had more carelessly picked a discretisation scheme we would expect to break all the symmetry. We can imagine a space of discrete theories around the Yang-Mills theory for which symmetry is lost at almost all points. The symmetric continuum theory exists at a critical point in this space. As the critical point is approached correlation lengths grow and details of the discretisation are lost. Symmetries are perfectly restored in the limit, and details of all the different discretisations are washed out.

If this is the case then it seems that the critical point is surrounded by a very high density of points in the space of theories. This is exactly what we would expect if universal behaviour dominating in theory space was to exhibit high symmetry. It also suggests that a dominant theory could be reformulated in many equivalent ways without any one particular formulation being evidently more fundamentally correct than another. Perhaps ultimately there is an explanation for the unreasonable effectiveness of mathematics in physics contained in this philosophy. If physics springs in such a fashion from all of mathematics then it seems likely that discovery of these laws will answer many old mathematical puzzles. Other similar arguments about origins of symmetry can be found in the work of Nielsen et al [129, 130].

The reader should be alerted to the fact that these arguments are at best incomplete. The aim is to present a philosophical viewpoint which enables us to see that there could be some fundamental principle from which the laws of physics are derivable. If we except this heuristic argument then there are two ways we can proceed. We could start by analysing the theory space of very general complex mathematical systems in an attempt to find the universal behaviour which dominates, or alternatively we can look for

possible pregeometries of space-time with high symmetry. In any case the Theory of Theories philosophy helps to widen our horizons. It seems appropriate to leave behind the continuum nature of space-time in the search for something more fundamental.

A Taxonomy of Pregeometries

Those who choose to pursue Wheeler's idea that the geometry of space-time must be replaced with a more fundamental pregeometry are faced with a difficult task. They can not bank on any direct guidance from experimental results if the pregeometry structure reveals itself only at Planck scales. If discreteness is an aspect of quantum gravity then we should be looking for models which allow curvature of space-time. The quantised space-time and regular lattice models are simply too rigid for this purpose. What then should we choose as our guiding star to lead us towards a good theory.

There are a number of ways the problem might be approached theoretically. We could study the properties of string theory, canonical quantisation of gravity and the quantum gravity aspects of black holes in the hope that this would lead us to a mathematical discovery which reveals the true nature of space-time to us. This is certainly being done by some of the best theoreticians in the business and has offered many important clues. A second approach for the more philosophically inclined is to try and deduce physics from basic principles such as a theory of theories or algorithmic information theory. Perhaps that is too ambitious since it has rarely proven to be the case in the past that physical law shares our philosophical preferences.

We might instead start to look at a variety of possible mathematical models in the hope that some model will be found to have properties which fit in with what we think we are looking for. Perhaps we can take the best ideas which have been tried in the past and crossbreed them in the hope of forming better models which combine the best qualities of their parents. There have indeed been many speculative models described in the literature and it is one of the aims of this paper to provide a large bibliography of references which the reader might use as a kind of gene pool for which to breed new models. The fact that many models have common features some of which are mathematically quite sophisticated suggests that it may really be possible to put together a theory which works.

To be a little more systematic it might be useful to make a list of some of the properties and features of space-time and dynamics in conventional

physics theories. Some of these will need to be discarded in the formulation of our pregeometry models and we can then feel encouraged if they re-emerge as a dynamically induced property of the theory either in an exact or approximate form.

Continuity: A feature of space-time in both general relativity and particle physics is its continuity. Space-time is modelled as a manifold with continuous coordinates. We have already argued that space-time may be discrete in some sense. Should we then start from a model without continuous coordinates and look for them to reappear in some form? Coordinates certainly have an artificial flavour even in the mathematical description of a manifold. Does this mean they have to be replaced with some discrete structure or should the discreteness be derived from a continuous model in the same sense that an atom has discrete energy levels which can be derived from the Shrodinger equation?

Dimension: Physicists have from time to time tried to answer the question as to why space-time has 3+1 dimensions [131, 132, 133, 135, 134]. It is well known that supergravity and superstring theories are most consistent in 10 dimensions and that the observed number of dimensions must come about through compactification of 6 spatial dimensions if such a theory is correct. Some authors consider the possibility that the number of dimensions could actually change in a phase transition at high energies [136, 137, 138, 99, 139]. If space-time has a fractal structure then the number of dimensions can be variable and non-integer [140, 141]. Should we therefore abandon dimensionality as fundamental altogether and start from a model which has no pre-set dimension? If space-time is discrete then it is possible that the number of dimensions can be derived dynamically and a number of pregeometry models of this type have been attempted [142, 143, 144, 145, 146, 147, 148, 149, 150, 151].

Metric: The space-time metric is the fundamental dynamical field in Einstein's formulation of general relativity and so one approach to generalising space-time is to look at more general geometric spaces which have a distance function defined on them [152, 148, 153]. General Relativity can, however, be reformulated in alternative ways making the metric less fundamental and already in string theory the metric tensor is just an aspect of a spin-two field which is a consequence of the dynamics. It may therefore be quite natural to discard the metric as a fundamental feature. Sometimes the term "pregeometric" is used to describe field theory defined on continuous manifolds without a metric structure, including Topological Field Theory [154, 155, 156, 157, 158, 159]. Many of the more interesting pregeometry

models encountered in this review discard both the metric and continuity

Causality: Loss of fundamental causality in a pregeometric model is difficult for some physicists to come to terms with. One pregeometric approach is to treat causality as the most important feature of the theory. This is typified by models of physics on causal structures [60, 160, 161, 162, 163, 147, 164]. On the other hand some physicists do not see causality as a fundamental part of physics but rather as something which is guaranteed only by dynamical effects as in Hawking's Chronological Protection Conjecture [165]. Causality could be violated at microscopic scales or even macroscopically and whether or not it is might perhaps be better left as a consequence of the theory rather than a fundamental principle. The most dramatic example of causality violation may be the Big Bang if it came about without any cause.

Topology: Wheeler emphasised the importance of topology in the small scale structure of space-time [166]. There are close ties between causality and topology on space-time since breakdown of causality may be linked to closed time-like curves and wormholes. A number of important papers have studied how to derive topology from discrete structures such as graphs or partially ordered sets (posets) [167, 153, 146, 168, 169, 170, 171, 172]. It is possible that a pregeometry may have no exact representation of topology and that it only arises dynamically or perhaps in a way which is less direct. An example of topological theory which may be connected with quantum gravity and space-time topology is knot theory [173, 174, 175, 176, 177]. Knot theory is automatically brought into physics whenever quantum groups are used and it is possibly through higher-dimensional algebras that topology may be understood in pregeometry [178].

Symmetry: The importance of symmetry in physics is often stressed. Much theoretical research has proceeded on the assumption that the symmetries so far observed in nature are just part of a larger symmetry, most of which is hidden by spontaneous symmetry breaking at low energy. Despite this, very few discrete pregeometries have been able to reflect space-time symmetries in a discrete form and some have suggested that such symmetry is not fundamental or even not exact [179, 56]. There are, however, a small number of possible approaches which do consider space-time symmetries in a discrete framework: Lattice Topological Field Theories [180, 181], Event-symmetric Space-time, [150] and non-commutative geometries with quantum group symmetry [182, 183].

Internal gauge symmetry can be more easily represented in discrete form on a lattice as was demonstrated by Wilson [47]. The key to finding a

successful pregeometry theory may be to formulate a unification of space-time and internal symmetries even if it is at the cost of causality, continuity and topology. It should also be born in mind that the concept of symmetry may have to be generalised from group theory to include quantum groups and possibly other more general algebras from category theory [185, 178]. Ultimately symmetry may be understood as a consequence of something more fundamental. See for example [184, 130].

Quantum Mechanics: There are some researchers who are strongly motivated by the desire to replace quantum mechanics with something else. Usually they would prefer a deterministic theory. Despite all the effort and many years of debate nobody has produced an experiment which is not in agreement with the quantum theory and most physicists regard its interpretation as being at least separated from physical dynamics.

But what form should quantum mechanics take in pregeometry? The answer depends on what kind of pregeometry you have and on what formulation of quantum mechanics you prefer. The main choice which must be made is whether to start from the canonical (Hamiltonian) formulation of quantum mechanics or from the path integral (Lagrangian) formulation [17]. Other possibilities exist such as a purely algebraic construction of an S-matrix.

The major difficulty with the canonical formulation is that it normally assumes a continuous time variable. If time is discrete or not defined at all then some modification of the procedure is needed [186]. However, with the canonical approach we at least get a well defined notion of state and can control unitarity.

If the canonical approach cannot be used or if a symmetrical treatment of time and space is desired then the path integral approach is better suited. A sum over histories can be defined in a very general context provided it is only desired to define some generalisation of the partition function or Greens functions. If we wish to understand the concept of state or calculate transition amplitudes then we are again in trouble. With the path integral approach we can postpone questions of causality and unitarity till later or they can be tackled with some generalised formalism such as that of Hartle [187].

For each of the above, a decision must be made as to whether it should be regarded as a fundamental feature of the pregeometry or a dynamically derived property with possibly only approximate validity. It seems to be mostly a matter of taste which governs the course taken by each researcher but at least it should be a consciously made choice. Perhaps there is not

really a right or wrong approach. It is quite possible that different models based on different principles can turn out to be equivalent.

Once it has been decided which properties of space-time are to be discarded and which are to remain fundamental, the next step is to choose the appropriate geometric structures from which the pregeometry is to be built. An obvious candidate for a fundamental building block is the space-time event. A continuous space-time manifold is itself a set of events with certain other structures built on top. It would be natural to retain the set of events and then replace the structures with something else. Wheeler described this as a "bucket of dust" [188, 1].

The space-time event does indeed appear as fundamental in many pregeometry models together with various other structures. E.g. a random graph can be composed of events and links which randomly connect them. The importance of events is also emphasised in some attempts to resolve the quantum measurement problem [189, 190, 191, 192] and this field sometimes overlaps with pregeometry models.

The causal net and poset models also use events as basic structures. Other models are built on cell complexes [144, 145, 193, 194] or simplicial complexes [195] in which the event is just a special case of a simplex. An event might also be a special case of a discrete string.

Sometimes the geometric approach is abandoned in favour of logic [60, 196, 169, 197] or algebra [198, 199, 200, 201]. Given a linear algebra we may take a basis of the algebra and associate each component with a physical object such as an event, simplex or string. However, a choice of basis is not unique and is only secondary to the algebraic structure, so the reality of those objects is subjective. There may be a group of transformations which generate changes of basis under which the theory is invariant. Perhaps physical objects become real only through spontaneous breaking of such symmetries.

An algebraic approach which has received a great deal of attention is non-commutative geometry [182, 183, 202]. The topological structure of space-time can be understood in terms of the differential algebra of functions on the manifold. According to Connes it is natural to generalise space-time by using other commutative algebras or non-commutative algebras and differential forms. The differential operator d has the property that $d^2 = 0$. This is an algebraic embodiment of Wheelers boundary of a boundary principle [120].

If the differential calculus on the product of a space-time manifold and a discrete space is constructed, it is possible to recover the standard model of electro-weak interaction with the Higgs field appearing as the connection

on the discrete space [203, 204] and more general unified models can be constructed from different spaces [205]. There are by now many references on this subject.

An obvious generalisation as a way to unify the particle forces with gravity is to look at noncommutative geometries or differential calculus on discrete sets or groups which represent the events and symmetries of space-time [206, 207, 208]. There is an important correspondence between such geometries and topological pregeometries formulated on posets [209, 210]

Finally, the importance of spin-structure in pregeometry must be noted. Wheeler points out that if the topology of space-time is non-trivial then spin-1/2 fields must be accounted for [166, 1]. A number of pregeometry models have used spin structure in inventive ways [66, 211, 212, 213].

For the remainder of this review I will look at what some of the major approaches to quantum gravity have to tell us about the nature of space-time at small distances.

Lattice Models of Gravity

Following the success of perturbative quantum field theory when applied to the electromagnetic and nuclear interactions it was natural for particle physicists to try the same approach to quantising gravity. The result was a disastrous theory in which covariance was lost and renormalisability could not be achieved [214, 215]. Quantum Gravity research remained at this dead end for some time until it was realised that gravity was somewhat different from other forces. Weinberg has shown that quantum gravity could be finite if it has a suitable ultraviolet fixed point [216].

QCD, the theory of the strong interaction can be formulated and analysed using Lattice Gauge Theories. The Wilson plaquette loop action for QCD [47] on a lattice formulated in the Euclidean sector has a certain elegance since it preserves a discrete version of gauge invariance and uses a group representation rather than a representation of a Lie algebra. Translation invariance is also preserved in a discrete form but the rotation group is only coarsely represented.

It is natural to ask whether a useful non-perturbative formulation of quantum gravity can be found on a lattice as it can for QCD. The immediate objection to this is that a lattice theory cannot reflect the important symmetries of space-time in a discrete form. It seems that they can only be recovered in the continuum limit.

It is common practice in Lattice Gauge Theories to convert the quantum theory into a 4 dimensional Euclidean statistical physics by a Wick rotation from real time to imaginary time. In the Euclidean sector numerical studies are feasible in Monte Carlo simulations.

Lattice studies of gravity are likewise made typically in a Riemannian sector where the metric has an all positive signature. A Riemannian theory of gravity would, however, present some interpretation problems since it is not possible to simply apply a Wick Rotation as can be done for the Euclidean sector of lattice theories without gravity. Nevertheless, if it could be shown that there was a continuum limit this would be strong evidence for the existence of a quantum gravity theory without the need to extend the classical theory.

Lattice studies of pure gravity start from the Regge Calculus [195] in which space-time is “triangulated” into a simplicial complex. The dynamical variables are the edge lengths of the simplices. In 4 dimensions an action which reduces to the usual Einstein Hilbert action in the continuum limit can be defined as a sum over hinges in terms of facet areas A_h and deficit angles δ_h which can be expressed in terms of the edge lengths.

$$S = \sum_h k A_h \delta_h \quad (1)$$

The model can be studied as a quantised system and this approach has had some limited success in numerical studies [217, 218, 219, 220].

One way to retain a form of diffeomorphism invariance is to use a random lattice on space-time instead of a regular lattice [221, 222]. A random lattice does not prefer any direction and Poincare invariance can be exact. An interesting aspect of the use of random lattices for quantum gravity is that if the lattice is allowed to change dynamically in some probabilistic fashion then its fluctuations merge with the quantum mechanics in the Riemannian sector. An action with fixed edge lengths but random triangulations [223] is given by,

$$S = -\kappa_4 N_4 + \kappa_0 N_0 \quad (2)$$

The partition function is formed from a sum over all possible triangulations of the four-sphere. N_4 is the number of four simplices in the triangulation and N_0 is the number of vertices. The constant κ_4 is essentially the cosmological constant while κ_0 is the gravitational coupling constant. Random triangulations of space-time appear to work somewhat better than the Regge Calculus with a fixed triangulation.

4 dimensions is qualitatively different from 2 or 3 dimensions where there can be no gravitational waves and hence no gravitons. In 4 dimensions there is evidence from numerical simulations that there is a second order phase transition under variation of the gravitational coupling constant and that a continuum limit with the correct Hausdorff dimension can be found at the critical point [224]. There is some debate about whether the number of triangulations is exponentially bounded [225, 226, 227]. This is an important condition for the model to be well defined in the large volume limit. It will probably require numerical studies on quite large lattices to settle this issue.

The question of the exponential bound is extremely important in quantum gravity. If it is not there then that might be interpreted as evidence that topology is important in the microscopic view of space-time. The models of Numerical triangulations studied numerically are restricted to a simple closed sphere topology. If this needs to be extended to a sum over other topologies to include contributions of wormholes and the like then it is important to know what weight should be given to each topology.

Even if dynamical triangulations provide a useful calculation method for quantum gravity there are some unanswered questions concerning its suitability as a fundamental formulation, among these is the question of ergodicity or computability [228].

String Theories

Despite the lack of experimental data above the Electro-Weak energy scale, the search for unified theories of particle physics beyond the standard model has yielded many mathematical results based purely on constraints of high symmetry, renormalisability and cancellation of anomalies. In particular, space-time supersymmetry [229] has been found to improve perturbative behaviour and to bring the gravitational force into particle physics. One ambitious but popular line of quantum gravity research is superstring theory [230, 231, 232]. String models were originally constructed in perturbative form and were found to be finite at each order [233] but incomplete in the sense that the perturbative series were not Borel summable [234].

Despite this there has been a huge amount of interest in a number of super-string theories and in the Heterotic String in particular [235]. The fact that this theory has an almost unique formulation with the interesting gauge group $E_8 \otimes E_8$ persuades many that it is the sought after unified field theory despite the fact that it is only finite in ten dimensional space-time. In 1926

Klein [236] proposed that a 5 dimensional theory due to Kaluza [237] could make physical sense if one of the dimensions was compactified. Kaluza-Klein theory has been applied to the heterotic string theory for which it has been shown [238] that six of the ten dimensions could be spontaneously compactified on a Calabi-Yau manifold or orbifold. This leaves an E_6 gauge group with suitable chirality modes just big enough to accommodate low energy particle physics. The difficulty which remains is that there are many topologically different ways the compactification could happen and there is no known way of picking the right one. To solve this problem it is thought necessary to find some non-perturbative analysis of the string theories. As a first step it might be necessary to construct a second quantised covariant String Field Theory [239].

There has been some preliminary success in formulating both open [240] and closed [241] bosonic String Field Theories. There have also been some important steps taken towards background independent formulations of these theories [125, 242]. However, they still fail to provide an explicit unification of space time diffeomorphism symmetry with the internal gauge symmetry. This is a significant failure because string theories are supposed to unify gravity with the other gauge forces and there is evidence that string theory does include such a unification [243]. Furthermore these formulations have not yet been extended to superstring theories and this appears to be a very difficult problem [244].

Conventional wisdom among the pioneers of string theories was that there is a unique string theory which is self consistent and which explains all physics. This view was gradually tempered by the discovery of a variety of different string theories but recently the belief in uniqueness has been reaffirmed with the discovery that there are hierarchies in which some string theories can be seen as contained within others [245, 246, 247]. This inspires a search for a universal string theory [248, 249, 250].

A successful theory of Quantum Gravity should describe physics at the Planck scale [14]. It is likely that there is a phase transition in string theories at their Hagedorn temperature near $kT = \text{Planck Energy}$ [251]. It has been speculated that above this temperature there are fewer degrees of freedom and a restoration of a much larger symmetry [252, 253, 254, 255]. This phase is sometimes known as the topological phase because it is believed that a Topological Quantum Field Theory may describe it. A fundamental formulation of string theory would be a model in which the large symmetry is explicit. It would reduce to the known formulations after spontaneous symmetry breaking below the Hagedorn Temperature.

One interpretation of the present state of string theories is that it lacks a geometric foundation and that this is an obstacle to finding its most natural formulation. It is possible that our concept of space-time will have to be generalised to some form of “stringy space” in which its full symmetry is manifest. Such space-time must be dynamical and capable of undergoing topological or even dimensional changes [256, 257, 258]. To understand stringy space it is almost certainly necessary to identify the symmetry which is restored at high temperature.

The notion that string theory has a minimum length is well established as a result of target space duality which provides a transformation from distances R to distances α/R where $\sqrt{\alpha}$ is the size of compactified dimensions at the Planck scale. A minimum length does not necessarily imply discrete space-time but it is suggestive.

Thorn has argued that large N matrix models lead to an interpretation of string theories as composed of pointlike partons [330, 259, 260]. A similar view has been pursued by Susskind as a resolution of paradoxes concerning Lorentz Contraction at high boosts and the black hole information loss puzzle [261, 94]. It is possible to calculate exact string amplitudes from a lattice theory with a non-zero spacing [80].

There are some remarkable features about these discrete string models. Firstly it is found that when the spacing between discrete partons is reduced below a certain limit there is a phase transition beyond which results coincide exactly with continuous models [80, 262, 263]. Even more odd is the apparent generation of an extra dimension of space so that models in 2+1 dimensions could become theories of 3+1 dimensions [264, 330, 265]. This has also been seen as an aspect of the continuum theory [266] and it seems that these concepts are not inconsistent with the algebraic construction of Topological Quantum Field Theory [267].

Perhaps the fact that string theory is finite to each order in perturbation theory is itself an indication that string theory is discrete. In lattice theories the renormalisation group is used to send the lattice spacing to zero but in string theory the coupling is not renormalised.

If a string is to be regarded as made up of discrete partons then it might make sense to consider the statistics of each parton. In the two dimensional worldsheet of the string a parton could have fractional statistics. If string partons are such that an increasing number of them are seen in a string at higher energies it may be necessary for the statistics to be divided up into fractions of ordinary fermionic or bosonic statistics. In the higher dimensional target space only half integer multiple statistics are permitted

to be observed.

Heuristically we might picture the string as an object consisting of n partons each with an interchange phase factor q such that q^n is real, i.e. 1 or -1 . This suggests that a continuum limit might exist where $n \rightarrow \infty$ on the worldsheet while the string has discrete aspects in target space. Such a model might be based on quantum group symmetries. There are already some encouraging results which suggest that it might be possible to formulate fractional superstring models [268].

This section would not be complete without referring to a number of other attempts to understand discrete string theory [269, 270, 271, 272, 273].

Canonical Quantisation of Gravity

There has also been some progress in attempts to quantise Einstein Gravity [274, 275] by canonical methods. A reformulation of the classical theory in which the connection takes the primary dynamic role instead of the metric [276, 277] has led to the Loop Representation of Quantum Gravity [173, 278] in which knot theory plays a central role. The fact that Einstein Gravity is non-renormalisable is considered to be not necessarily disastrous since gravity theories in 1+1 and 2+1 dimensions have been successfully quantised by various means [279]. In the latest versions of the theory the quantum gravity states are based on Penrose spin networks [280].

String theory originated as a proposed theory of the strong interactions before being replaced by QCD. It is possible that QCD may still be possible to reformulate as a string theory at least approximately if not exactly. The loop representation also first appeared in connection with Yang Mills theories like QCD. [281]

Another likeness between the loop representation and string theories is that there are attempts to understand them in terms of field variables and groups defined on loop objects [282]. This and other similarities may be more than superficial [283]. Superstring theory and the Loop Representation of quantum gravity in the forms we know them can not be equivalent since the former only works in ten or eleven dimensions while the latter only works in three or four. It is possible that they could be different phases of the same pre-theory provided that pre-theory allows changes of dimension. Alternatively it could just be that we just have not learnt how to do string theory in 4 dimensions and the loop representation in 10 dimensional superspace yet, or more likely that the role of dimension has been so far

misunderstood.

A major difference between the canonical quantisation and string theory is that it is an attempt to quantise gravity in the absence of matter fields. The view from Canonical Quantisation supports that of the lattice quantum gravity schemes that a theory of quantum gravity without matter exists but string theorists often express the belief that gravity can only be quantised when other fields are included.

If quantum gravity was an easy business then somebody would have found the formalism which allows you to form the perturbative expansion of the loop representation about a fixed background and demonstrate its equivalence to string theory. Differences between the two suggest that their relationship may not be so simple and in fact the loop representation has at least as many consistency problems to resolve as string theory [284] before such a program might be realised.

Nevertheless the program is developing rapidly and the one of its great strengths is that the form of the theory is derived directly from canonical quantisation methods of gravity rather than being constructed ad hoc. The fact that this leads to a discrete spectrum for volume and area operators [83] is a powerful argument in favour of discrete aspects of space-time in quantum gravity. In one version the theory is formulated on a discrete lattice without losing diffeomorphism invariance enabling a convenient calculation scheme [285].

Canonical quantisation of gravity may allow many other useful observations on the nature of space-time to be made.

Quantised Space-Time

One of the most important principles in modern theoretical physics is that of symmetry. It is quite likely that the observed symmetries in nature are the remnants of a much larger symmetry which existed at the beginning of the universe but which were successively broken down to smaller and smaller symmetries as the universe expanded cooled. Knowing the full original symmetry is the key to knowing the laws of physics.

Traditionally the classification of symmetry was regarded as being equivalent to the classification of abstract groups. This view changed with the discovery of supersymmetry which allows us to define symmetries between fermionic and bosonic particles but which is not related to a classical group. In the last decade a new type of symmetry has been discovered and widely

researched. This symmetry is related to an algebraic object which is a deformation of the classical notion of group. It is known as a quantum group.

Quantum Groups first arose in the context of exactly solvable 1+1 dimensional lattice systems [286] and the quantum inverse scattering method [287]. Drinfeld and Jimbo identified the relevant algebraic structure as a quasi-triangular Hopf algebra [288, 289, 290]. The same structure was discovered independently by Woronowicz in the context of non-commutative geometries [291, 292]. Since then quantum groups have been recognised as important symmetries in many different types of physical system and a large number of technical papers have been written. In particular it is expected that quantum groups will have a major role to play in theories of quantum gravity.

Quantum groups are an example of what mathematicians would call a deformation. A structure is defined which has a dependency on a complex parameter q . In the special case $q = 1$ the structure corresponds to a group. In general it is not a group but still has many of the properties that make groups useful in physics.

Other types of deformation have already proved useful to physicists. Quantum mechanics is a particular example which has a deformation parameter given by Planck's constant \hbar . In the $\hbar \rightarrow 0$ limit we reach classical physics. Quantum groups may allow a new deformation of physics which brings in a minimum length scale. It is possible that this could lead to a deformed version of general relativity which is finite when quantised. Another reason to suppose that quantum groups are important in quantum gravity is the relationship between quantum groups and knot invariants [176]. Knot theory is known to be relevant to quantum gravity as topological field theory [173].

For a detailed description of how a group can be deformed as a Hopf algebra there are many references which can be consulted [293, 294, 295, 296, 297, 298].

The concept of space-time quantisation goes back to the 1940's when Snyder proposed that space-time coordinates should be replaced by non-commutative operators [19, 20]. The aim was to introduce a fundamental length into physics in order to avoid divergences which plagued quantum field theory. (added note: The reader should consult Sec. 1.3 of *Principles of Quantum General Relativity* by E. Prugovecki [32] for even earlier references to papers on quantised space-time by Ruark, Flint, Richardson, Firth, Landau, Peierls, Glaser, and Sitte.) Snyders model failed because although it was Lorentz invariant it destroyed translational invariance. Yang proposed

a model on anti-de Sitter space-time which also had translation invariance [21]. It was unphysical because it implied a curvature of space-time on the scale of the Planck length rather than the scale of the universe. Townsend proposed a theory in which this could be realised by gauging the de Sitter group in place of the Poincare group [299].

The problem of describing a discrete space-time which has an adequate discretisation of the full Poincare group has always been an obstacle to constructing field theory with a fundamental minimum length scale. Schild deduced that an integer lattice could preserve a large subgroup of the Lorentz group but one which was far too coarse [300]. Hill argued that a space-time with rational coordinates solved the problem [57] but it is debatable whether such a dense covering can be described as discrete.

There are indeed some models of space-time which have some form of space-time symmetry and a minimum length scale. One possibility is to use a random lattice or dynamical triangulations. In this case symmetry is regained after quantisation which includes a sum over all triangulations while keeping a minimum length scale. Again an event symmetric space-time also has a discrete version of diffeomorphism invariance. We can also mention Lattice Topological Field Theory as another example. These possibilities are described in other sections.

We may learn something useful if we can formulate a theory of discrete space-time on flat space. The discovery of quantum groups has brought about a revival of quantised space-time. If our understanding of symmetry is broadened to include quantum group symmetries then we can use q -deformed Lorentz [301] or Poincare groups [302, 303]. By factoring out the Lorentz algebra it is possible to define a deformation of Minkowski space. Deformations of Euclidean spaces are equally possible and worthy of study [304]. Many of these models have a discrete differential calculus.

These spaces and groups are not defined directly. They are defined in terms of an algebra of functions on the spaces. It is possible to construct differential algebras which act on these functions and it is found that these derivatives are finite difference operators. In this sense we have succeeded in constructing discrete spaces without abandoning symmetry principles.

The next step would be to construct field theories on these spaces. It has been found that there are important constraints on which gauge theories can be constructed [305, 306, 307, 308, 309, 310]. It might also be interesting to try to gauge the symmetry in the same sense as in gravity and supergravity theories [311]. If the flat model is discrete then the gauged model which would describe curved space-time could be especially interesting.

A limitation of quantum groups as a generalisation of symmetry is that they are not general enough to include supersymmetry. The universal enveloping algebra of a Lie superalgebra is not a Hopf algebra. It is a super-Hopf algebra. An alternative structure developed by Majid is the braided group which does include supersymmetry. Braided groups are related to quantum groups but perhaps the most general algebraic structure to describe symmetry has not yet been defined. Majid also uses his techniques to quantise space-time [312].

There are many papers written on these and related subjects and it should also be noted that quantum groups and other related algebraic structures are also of principle importance in string theory and canonical quantisation of gravity.

Topological Quantum Field Theories

A criticism often made against the way superstring theories have been developed is that they are not explicitly covariant or background free. In contrast Witten introduced the concept of Topological Field Theories and represented gravity in three dimensions with the Chern-Simons-Witten (CSW) model [313]. This turned out to have as many useful results in topology as it has in physics. Atiya considered how this would extend to quantum theories and produced a set of axioms describing properties a quantum gravity should have as a Topological Quantum Field Theory [175].

A surprising series of discoveries which led to an alternative understanding of Topological Field Theories had begun years before when it was found that the Regge Calculus [195] in three dimensions could be approximated by a formula involving $6j$ -symbols from the quantum theory of angular momentum described by representations of the group $SU(2)$ [211]. The Ponzano-Regge model is constructed from tetrahedral simplices having edge lengths which are quantised to half integer values. These j -numbers have a dual interpretation as either spins or lengths. The relationship with Penrose spin-networks [66] was also studied [314].

The Ponzano-Regge model has now been reformulated as the Turaev-Viro model with $SU(2)$ replaced by the quantum group $U_q(su(2))$ at q an r -th root of unity [315]. This provides a natural regularisation of the model with the lengths limited to less than $r/2$. The original formulation is recovered in the $q \rightarrow 1$ limit. They showed that the partition function was independent of the triangulation and could therefore only depend on the topology of

the triangulated space. The model is therefore a topological quantum field theory and can be regarded as a successful quantisation of 3 dimensional gravity.

It is now known that the Ponzano-Regge model is equivalent to a Chern-Simons-Witten model with gauge group $ISO(3)$ [316, 317] and can also be transformed to a loop representation [318]. Cherns-Simons models can be related to string theory [319] and the Turaev-Viro model can be reformulated in terms of surfaces (string worldsheets) [320] so within this 3-dimensional model we already see a unification of many of the main-stream ideas in quantum gravity.

The equivalence between simplicial gravity and topological quantum field theories could be interpreted as a partial resolution of the discrete-continuous dual nature of space-time but some care is needed. In this duality the discreteness does not appear at just the Planck scale unless space-time is assumed to be topologically complex at that scale in the sense of the space-time foam of wormhole geometrodynamics [321, 322]. In a topologically simple space-time the triangulation can be made so coarse that there would be only a few degrees of freedom left so there is a discreteness even at large scales. Furthermore it must be appreciated that three dimensional gravity is very simple compared to four dimensional gravity, there are no local excitations or gravity waves because there are no Weyl tensor components in less than four dimensions.

One further formulation of simplicial 3D gravity due to Boulatov is of particular interest. This starts from a perturbation theory defined on a field of triangular objects moving on a quantum group. The triangular objects are made to interact through a tetrahedron vertex. The perturbation expansion in the coupling constant of this pre-theory is then equivalent to the Turaev-Viro model [323]. This is of special interest because it is a pre-theory in which a 3 dimensional space-time is dynamically generated and also because the interactions of the triangular objects are reminiscent of string vertices in string field theory.

Given the success of this approach in 3 dimensions it is natural to try and generalise to 4 dimensions and indeed several people have pursued this line [324, 325, 326]. Sadly these models have turned out to be too simple to be theories of quantum gravity so far [327, 328]. A more speculative proposal is that 4 dimensional physics can be found in the 3D simplicial models where tetrahedral inequalities are violated [329]. Such an approach would be consistent with the hologram ideas of Thorn, t'Hooft and Susskind [330, 93, 94]. Another similar possibility is that the 3D TQFT could be

considered as a state of 4D gravity from which time evolution could be inferred [331]. Crane has been developing these ideas further in order to try to understand the physical aspects to topological quantum field theories [267].

Event-Symmetric Space-Time

Finally I turn to my own research on space-time structure.

My belief is that the symmetry so far discovered in nature is just the tiny tip of a very large iceberg most of which is hidden beneath a sea of symmetry breaking. With the pregeometric theory of *event-symmetric physics* I hope to unify the symmetry of space-time and internal gauge symmetry into one huge symmetry. I hope that it may be possible to go even further than this. Through dualities of the type being studied in string theory it may be possible to include the permutation symmetry under exchange of identical particles into the same unified structure.

The theory of Event-Symmetric space-time is a discrete approach to quantum gravity [150]. The exact nature of space-time in this scheme will only become apparent in the solution. Even the number of space-time dimensions is not set by the formulation and must be a dynamic result. It is possible that space-time will preserve a discrete nature at very small length scales. Quantum mechanics must be reduced to a minimal form. The objective is to find a statistical or quantum definition of a partition function which reproduces a unified formulation of known and hypothesised symmetries in physics and then worry about states, observables and causality later.

Suppose we seek to formulate a lattice theory of gravity in which diffeomorphism invariance takes a simple and explicit discrete form. At first glance it would seem that only translational invariance can be adequately represented in a discrete form on a regular lattice. Dynamical triangulation is much better but still the symmetry is not explicit and only appears after quantisation. This overlooks the most natural generalisation of diffeomorphism invariance in a discrete system.

Diffeomorphism invariance requires that the action should be symmetric under any differentiable 1-1 mapping on a D dimensional manifold M_D . This is represented by the diffeomorphism group $diff(M_D)$. On a discrete space we could demand that the action is symmetric under any permutation of the discrete space-time events ignoring continuity altogether. Generally I use the term *Event-Symmetric* whenever an action has an invariance un-

der the Symmetric Group $S(\mathcal{U})$ over a large or infinite set of “events” \mathcal{U} . The symmetric group is the group of all possible 1-1 mappings on the set of events with function composition as the group multiplication. The cardinality of events on a manifold of any number of dimensions is \aleph_1 . The number of dimensions and the topology of the manifold is lost in an event-symmetric model since the symmetric groups for two sets of equal cardinality are isomorphic.

Event-symmetry is larger than the diffeomorphism invariance of continuum space-time.

$$\text{diff}(M_D) \subset S(M_D) \simeq S(\aleph_1) \quad (3)$$

If a continuum is to be restored then it seems that there must be a mechanism of spontaneous symmetry breaking in which event-symmetry is replaced by a residual diffeomorphism invariance. The mechanism will determine the number of dimensions of space. It is possible that a model could have several phases with different numbers of dimensions and may also have an unbroken event-symmetric phase. Strictly speaking we need to define what is meant by this type of symmetry breaking. This is difficult since there is no order parameter which can make a qualitative distinction between a broken and unbroken phase.

The symmetry breaking picture is not completely satisfactory because it suggests that one topology is singled out and all others discarded by the symmetry breaking mechanism but it would be preferred to retain the possibility of topology change in quantum gravity. It might be more accurate to say that the event-symmetry is not broken. This may not seem to correspond to observation but notice that diffeomorphism invariance of space-time is equally invident at laboratory scales. Only the Poincare invariance of space-time is easily seen. This is because transformations of the metric must be included to make physics symmetric under general coordinate changes. It is possible that some similar mechanism hides the event-symmetry.

It is possible to make an argument based on topology change that space-time *must* be taken as event-symmetric in Quantum Gravity. Wheeler was the first to suggest that topology changes might be a feature of quantum geometrodynamics [321]. Over the past few years the arguments in favour of topology change in quantum gravity have strengthened see e.g. [332]. If we then ask what is the correct symmetry group in a theory of quantum gravity under which the action is invariant, we must answer that it contains the diffeomorphism group $\text{diff}(M)$ for any manifold M which has a permitted topology. Diffeomorphism groups are very different for different topologies

and the only reasonable way to include $diff(M)$ for all M within one group is to extend the group to include the symmetric group $S(\aleph_1)$. There appears to be little other option unless the role of space-time symmetry is to be abandoned altogether.

It is unlikely that there would be any way to distinguish a space-time with an uncountable number of events from space-time with a dense covering of a countable number of events so it is acceptable to consider models in which the events can be labelled with positive integers. The symmetry group $S(\aleph_1)$ is replaced with $S(\aleph_0)$. In practice it may be necessary to regularise to a finite number of events N with an $S(N)$ symmetry and take the large N limit while scaling parameters of the model as functions of N .

Having abandoned diffeomorphisms we should ask whether there can remain any useful meaning of topology on a manifold. A positive answer is provided by considering discrete differential calculus on sets and finite groups [206].

In some of the more physically interesting models the symmetry appears as a sub-group of a larger symmetry such as the orthogonal group $O(N)$. It is also sufficient that the Alternating group $A(N)$ be a symmetry of the system since it contains a smaller symmetric group.

$$S(N) \subset A(2N) \tag{4}$$

The definition of the term event-symmetric could be relaxed to include systems with invariance under the action of a group which has a homomorphism onto $S(N)$. This would include, for example, the braid group $B(N)$ and, of course, quantum groups such as $SL_q(N)$.

Renormalisation and the continuum limit must also be considered but it is not clear what is necessary or desired as renormalisation behaviour. In asymptotically free quantum field theories with a lattice formulation such as QCD it is normally assumed that a continuum limit exists where the lattice spacing tends to zero as the renormalisation group is applied. In string theories, however, the theory is perturbatively finite and the continuum limit of a discrete model cannot be reached with the aid of renormalisation. It is possible that it is not necessary to have an infinite density of events in space-time to have a continuum or there may be some alternative way to reach it, via a q-deformed non-commutative geometry for example.

It stretches the imagination to believe that a simple event-symmetric model could be responsible for the creation of continuum space-time and the complexity of quantum gravity through symmetry breaking, however,

nature has provided some examples of similar mechanisms which may help us accept the plausibility of such a claim and provide a physical picture of what is going on.

Consider the way in which soap bubbles arise from a statistical physics model of molecular forces. The forces are functions of the relative positions and orientations of the soap and water molecules. The energy is a function symmetric in the exchange of any two molecules of the same kind. The system is consistent with the definition of event-symmetry since it is invariant under exchange of any two water or soap molecules and therefore has an $S(N) \otimes S(M)$ symmetry where N and M are the number of water and soap molecules. Under the right conditions the symmetry breaks spontaneously to leave a diffeomorphism invariance on a two dimensional manifold in which area of the bubble surface is minimised.

Events in the soap bubble model correspond to molecules rather than space-time points. Nevertheless, it is a perfect mathematical analogy of event-symmetric systems where the symmetry breaks in the Riemannian sector to leave diffeomorphism invariance in two dimensions as a residual symmetry. Indeed the model illustrates an analogy between events in event-symmetric space-time and identical particles in many-particle systems. The models considered further are more sophisticated than the molecular models. However, the analogy between particles and space-time events remains a useful one.

There are many possible event-symmetric models which can be constructed [150] but the most interesting ones must be the event-symmetric string theories [333]

It might be asked what status this approach affords to events themselves. Events are presented as fundamental entities almost like particles. Event orientated models are sometimes known as Whiteheadian [189] but Wheeler preferred to refer to a space-time viewed as a set of events without a geometric structure as a “bucket of dust” [188, 1]. In some of the models we will examine it appears as if events are quite real, perhaps even detectable. In other models they are more metaphysical and it is the symmetric group that is more fundamental. Indeed the group may only arise as a subgroup of a matrix group and the status of an event is then comparable to that of the component of a vector. Then again in the discrete string models we will see that events have the same status as strings.

Above all the event-symmetric approach seems to suggest a Machian view of physics. Space-time takes a secondary role to events which are identified with particles or sting states.

Discussion

Physicists and philosophers have been interested to the small scale structure of space-time for many years. There have been many papers written describing various models of pregeometries or quantised space-time and recently serious interest in such research has seen an explosion of activity.

There have always been suggestions that space-time must be discrete at small scales but the motivation for this assertion has changed with time. Initially the justification was largely meta-physical. The fact that measurement in physics does not reflect mathematical properties of real numbers or the existence of scale dependence of physical law could be cited as evidence.

The emergence of quantum theory led to speculations about space-time quantisation which were reinforced by the need to renormalise in quantum field theory. At the time many physicists found the procedure unsatisfactory and felt that field theory could only be consistent if there was a small scale cut-off. The measurement problem also added to the motivation to find something more fundamental which would manifest at small scales.

Many of these concerns have subsided but the theoretical evidence for a minimum length at the Planck scale in quantum gravity and constraints imposed by the black hole information loss paradox have taken their place as motivation for discrete theories of space-time.

To systematically characterise pregeometric models we have discussed a number of physical properties which might be either abandoned or taken as fundamental in a pregeometry. A variety of different models appear as a result. Some of the main classes can be summarised as follows,

- Quantised space-time
- Cellular Automata
- Lattice Field Theories
- Quantum metric spaces
- Causal nets
- Poset models
- Simplicial quantum gravity
- Topological Quantum Field Theory
- Field theory on a cell complex
- Non-Commutative geometry
- Event-symmetric space-time

In addition to these I have briefly examined the main approaches to quantum gravity which tell us a great deal about the nature of space-time at the smallest possible scales.

There are intricate relationships between these models which lend hope that a realistic model of space-time may be realisable despite the unlikelihood of direct help from experiment. Further clues on the right way to go continue to come from string theory and semi-classical studies of black holes. At the same time the emergence of new mathematical frameworks which generalise the classical notion of symmetry and uncover powerful relationships between algebra, topology and field theory is at last providing us with the tools to explore the small scale structure of space-time.

The construction of quantum groups has been absorbed into almost all approaches to Quantum Gravity as if it was a discovery well overdue. Discreteness of space-time is also universal but so is the importance of topology, a clear sign that a full theory of Quantum Gravity must resolve its discrete/continuum dual nature.

Most impressive is that old ideas such as Snyder's quantised space-time, Regge calculus, Penrose spin networks and Wheeler's pregeometry are now all proving to be prophetically relevant. It is a revelation of the power of human thought that this should be the case as well as a dramatic demonstration of the effectiveness of mathematics in physics. An observation which must have some profound explanation.

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